THE FOUNDATIONS OF MATHEMATICS

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APRIL 21, 2014

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In mathematics, truth is arguably the most essential of its components. Suppose then, that a person, Jack, goes outside and ‘counts’ an object. After he returns, Jill goes out and ‘counts’ an object as well. The question then: how many objects were counted? The natural response would be to say that they counted two objects, even though by doing so, one makes the assumption that two independent objects were counted. It further assumes that if they had counted separate objects, taking these objects together would result in having two objects.

The problem with this, is that even though it does appear to be rational, none of these notions (number, counting, etc.) have definitions to the layman without the use of analogy. Number is the basis for the entirety of mathematics, yet its definition eludes so many that it could be said to be axiomatic. Even Leibniz, who developed infinitesimal calculus simultaneously with Newton, “urged constantly that axioms ought to be proved and that all except a few fundamental notions ought to be defined” (Russell 5)

However, axiomatic objects given meaning within a system are difficult, or outright impossible, to prove or define within that same system. In this way, proving the validity of most mathematical axioms becomes an arduous task if undertaken using mathematics itself. There is, however, an alternative to be found in logic. To derive all of mathematics only requires an understanding of very few logical notions. Namely that of Implication, Propositional Functions, Classes, and Relations, each of which follows naturally from the former.

For the sake of brevity, the content contained herein will focus more on the qualitative aspects of the topics at hand in order for the information to be readily digestible to those who may not have any formal training in logic or its principles. The following will concern itself with the very limited approach to merely constructing a valid field for the class of natural numbers, with the intent of defining addition and verifying logically, that 1+1 is equal to 2. To work within a system of ideas, however, it is essential to know the definition of that system.

To begin, Bertrand Russell's definition of pure mathematics:

> Pure Mathematics is the class of all propositions of the form *"p* implies *q*," where *p* and *q* are propositions containing one or more variables, the same in the two propositions, and neither *p* nor *q* contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member […] the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics *uses* a notion which is not a constituent of the propositions with it considers, namely the notion of truth. (Russell 3)

Implication is defined in the Merriam-Webster dictionary as "involving as a consequence, corollary, or natural inference." Implication can be thought of as belonging to two categories: formal implication and material implication (or in modern logic: deductive and inductive reasoning, respectively). The most common and, more often than not, first formal implication beginning logic students are introduced to is the following: Socrates is a man. All men are mortal. Therefore, Socrates is mortal.

Formal implication is wholly concerned with the order and appearance of terms in a series of propositions. So long as the propositions themselves are true, the conclusion must then follow with absolute certainty. That is to say, it is not essential to concern oneself with the nature of the connection between the statements themselves (from the example: *x* is a man) and focus attention wholly on the terms being addressed (*Socrates*, *man,* and *mortal*).

Material Implication (hereafter referred to as induction) is concerned with the nature of the statements themselves. Inductive implication in its entirety is constructed from 4 axiomatic or primitive propositions. These are defined as follows: (1) The Contradictory Function – "[the] argument *p*, where *p* is any proposition, is the proposition which is contradictory of *p*, that is, the proposition asserting that *p* is not true" (Whitehead and Russell 1:6). The contradictory function is denoted by '∼p'. It is important to note, however, that contradictory and *contrary* do not have the same meaning. To clarify by analogy: The contrary of up is down. The *contradictory* of up, is every direction that is not up. Likewise, the contradictory of a proposition is everything that is not that proposition. (2) The Logical Sum – "[this] is a propositional function with two arguments *p* and *q*, and is the proposition asserting the *p* or *q* disjunctively, that is, asserting that at least one of the two *p* and *q* is true" (Whitehead and Russell 1:6). In simpler terms, the logical sum of two arguments is expressed in the phrase 'either *p* or *q* is true.' (3) Logical Product – "[this] is a propositional function with two arguments *p* and *q*, and is the proposition asserting *p* and *q* conjunctively, that is, asserting that both *p* and *q* are true" (Whitehead and Russell 1:7). Like the logical sum, this in plain English would be spoken as '*p* and *q* are both true'. (4) The Implicative Function – "[this] is a propositional function with two arguments *p* and *q* and is the proposition that either not-*p* or *q* is true, that is, if *p* is true, then *not-p* is false then […] the only alternative left by the proposition (∼*p* v *q*) is that *q* is true" (Whitehead and Russell 1:7). Though this may seem like an unnecessary proposition at first, this is the very basis for inductive implication. Reviewing the proposition, we can reword it from 'if *p* is true, then *not-p* is false and *q* is true' into the last, and arguably most important, fundamental proposition. By omitting '*not-p'* the proposition becomes: 'if *p* is true then *q* is true', or in other words, *p implies q*. This notion of 'implies' also has its own logical operator (⊃) and is written *'p* ⊃ *q*' (Read: If *p* then *q* or *p* implies *q*; these two expressions will be used interchangeably).

With these four basic logical functions it is possible to construct the remaining logical forms though it will not be necessary for the current discourse on mathematical foundations. It is merely necessary to have an understanding of the ideas of *not, or, and,* and *implies* within the context of logic. These principles operate only by the possession of a specific propositional function; which can be described as a proposition that contains within it some variable component (usually the subject) that can be replaced by any term. A *propositional function* will be formally defined as follows: "φ*x* is a propositional function if, for every value of *x*, φ*x* is a proposition, determinate when *x* is given" (Russell 19). In other words, a proposition is a propositional function when given some particular variable $\hat{x}$ it becomes a proposition.

An example may help to illustrate this concept. Consider the propositional function '*x* is a man' (hereafter abbreviated (Mx)), this in and of itself is not a proposition as it asserts nothing, given that the *x* contained within is not a proper subject but merely a placeholder, called an 'individual constant' in predicate logic. The assertion could then be made that given some particular $\hat{x}$ (which will be taken to mean an actual term that has simply not been defined) '*x* is a man' does indeed become a proposition. In this instance, we could replace *x* in (Mx) with Socrates, Plato, Aristotle, rock, etc. and all of these would be propositions, though not all of them would yield positive truth values.

It becomes necessary, then, to discuss how propositional functions are evaluated for truth or falsehood given some collection of particular variables, called the range of a propositional function. Formally, the range of a propositional function will be defined as "Corresponding to any propositional function φx, there is a range, or collection, of values, consisting of all the propositions (true or false) which can be obtained by giving every plausible determination to x in φx" (Whitehead and Russell 1:15-16). There are three possible cases with that a propositional function can be evaluated for either positive or negative truth values, one of which must be a true statement. Either (1) all propositions of the range are true, (2) some propositions of the range are true, or (3) no propositions of the range are true. As for denoting, (1) will be denoted by $\forall x|\varphi x$ (Read: for all *x* such that φ*x* is true). (2) Will be denoted by $\exists x|\varphi x$ (Read: there exists an *x* such that φ*x* is true). (3) will be denoted by $\forall x|{\sim}\varphi x$ (Read: for all *x* such that φ*x* is false, more simply, every value of *x* causes φ*x* to evaluate to a false truth value).

Further, a propositional function is said to be identical to another propositional function if $(\forall x|(\varphi x))\supset(\psi x)$. In other words, two propositional functions are identical if every value that makes the first propositional function true also implies the truth of another propositional function.

Special attention will be paid to the notion of *such that* which appears in the evaluation of propositional functions. "The values of *x* which render a propositional function φ*x* true are like the roots of an equation – indeed the latter are a particular case of the former – and we may consider all the values of *x* which are *such that* φ*x* is true," The collection of all of these specific individual constants forms what is called a *class*. This concept of *class* is critical to the construction of a definition for number. A class will be defined as 'all values of $\hat{x}$ that satisfy (or yield a positive truth value) for some propositional function ϕ*x*.' In other words, every value that causes a propositional function to be true forms a class. It is to be noted, however, that classes can be defined, within the construct, in one of two manners, either extensionally or intensionally. In order to discuss the difference between these two categorizations of classes we require definitions of the categories.

The extensional properties of the class are the properties which are required to be a member of that class. To state more formally, the extensional properties of a class are given by the *class concept*. A class concept can be described as the ambiguation of the property required to be a member of a particular class. In the running example 'Socrates is a man,' *man* in its ambiguous sense would be the *class concept* and the extensive property of the class. Therefore, to define a class extensively, requires enumeration of all possible objects that fall under the ambiguous category of *man.* This method of class definition is only practical for small finite sets, and does not suit the purpose of mathematics in general.

More appropriate would be the use of the intensional categorization. Contrary to the extensional definition of classes, the intensional definition gives the meaning of a term by specifying all the properties required to come to that definition. As an example, to belong to the class of *bachelor* a variable *x* must have the properties of *unmarried* and *male*. It is this method of class distinction that will be used in the construction of the natural numbers, as it allows for the inclusion of elements into a class without the need for enumeration of those elements. Instead all that is required is a set of restrictions as to which propositions a variable must satisfy to be included in a class.

An essential step is then to codify the various properties of classes. If some variable belongs to class of variables that satisfies some propositional function, that propositional function will have a positive truth value for any element of that class. This amounts to a more formal statement of the previously given definition. Following this, "If ϕ*x* and ψ*x* are equivalent propositions for all values of *x*, then the class of *x*'s such that ϕ*x* is true is identical with the class of *x*'s such that ψ*x*" (Russell 20). More simply, two classes are equivalent, if all of the elements of both classes satisfy the same propositional function (equivalence of propositional functions was defined previously, and it is this definition which is used to denote propositional function equivalence). The rest of the properties of classes follow from the fundamental propositions. As an example, the syllogism *p* implies *q, q* implies *r,* therefore *p* implies *r*, is also true of classes, which can be demonstrated with a simple substitution of *x* is an *a*, *x* is a *b*, and *x* is a *c*, (where *a, b,* and *c* are classes). Substituting them in to the syllogism produces *x* is an *a* implies *x* is a *b*, and *x* is a *b* implies *x* is a *c*, therefore *x* is an *a* implies *x* is a *c.*

An important subsequent of the concept of classes is the notion of relation. Formally this will be defined as "the class of couples (*x*, *y*) for which some given function ψ(*x, y*) is true. It's relation to $\psi(\hat{x}, \hat{y})$ will be just like that of the class to its determining function" (Whitehead and Russell 1:211) More simply, a relation between two variables is the propositional function which those two variables yield a positive truth value. A cogent example to help clarify the concept of a relation may be found within the proposition "Socrates is a man." In this proposition, there exists a relation of identity between 'Socrates' and 'man', that is, Socrates is related to man by identity. Relations, it would seem then, are almost wholly concerned with the verb contained within a proposition, which denotes the relation between the two terms.

With this concept and those previously discussed, it is possible to construct a working definition of number, with careful care taken to maintain generality in so much as it is possible within the confines of the available logical tools. However, any further exploration will be covered as the need arises.

When any class is given, there are a certain number of individuals to which this class is applicable, that is, the elements of the class that satisfy whatever propositional function the class is created by. This enumeration of the elements of a given class can be viewed as the property of a class which will be called the class's *number*. Further, two classes are regarded as *similar* when they contain the same number. The simplest place to start with a definition of number is by abstraction.

> The relation of similarity between classes has the three properties of being reflexive, symmetrical, and transitive; that is to say, if *u, v, w* be classes, *u* is similar to itself, if *u* be similar to *v* then *v* is similar to *u*; and if *u* be similar to *v*, and *v* be similar to *w*, then *u* is similar to *w*. […] Now [when] these three properties of a […] relation holds between two terms, those terms have a common property. This common property we call their number. (Russell 114)

There exists, however, a critical defect in this definition by abstraction. The fact that it has not defined the relation in such a way as to secure that the definition is satisfied by *only* one singular property (namely, the class of classes that contains that property which is *the* number of the classes). Instead, what is obtained is an entire class of this property, such that it may have a potentially infinite number of terms.

The classes in question must have a relation to one another such that the classes have an identical property of number, but must have a relation to the class containing all the classes of a given number that they share with no other class. That is the one-one relation, and the many-one relation. It is better for clarity to define these relations by ambiguity. Two classes have a one to one relation if, for every particular *x* in class *a*, it is correlated to one, and only one, *y* in class *b*. In other words, any one term of either corresponds to one and only one term of the other. A many-one relation works similarly, however, instead of correlating a term from one class to another, it is correlating all terms to the class of classes.

Using these two relations, the definition of number may be refined into a concept that satisfies all of the properties of number in mathematics. “Membership of this class of classes [the number] is a common property of all the similar classes and of no others; moreover every class of the set of similar classes has to the set a relation which it has to nothing else, and which every class has to its own set” (Russell 115).

At first, this notion of numbers as classes of classes may seem a bit odd and very counterintuitive. However an example ought to help clarify its workings. In the first place, such a word as *couple* obviously does denote a class of classes. Thus “two men” means there is a class of men that also belongs to the class of couples. It should be noted, however, that this definition is only concerned with real, positive, numbers.

From this definition of number then, it is fairly straightforward to create the arithmetic addition of classes of classes. In fact, the definition of addition has already been outlined in the primitive propositions, and as classes are collections of variables for which primitive propositions are true, it is possible then do define arithmetic addition by *logical* addition. A restatement of the logical sum, modified for classes: If *u* and *v* are classes, then their logical sum is *u* V *v*. Applying this to arithmetical addition: If *u* and *v* are mutually exclusive *numbers* (i.e. a class of classes) then their arithmetical sum will be the class of classes such that *u* V *v*. More simply, the logical sum of two classes (of number) will be the class that contains all the elements of both classes (of numbers).

Initially, this appears to fall apart when summation of identical numbers is considered, since the class that contains both elements of the classes being summed is the classes being summed. This requires the introduction of *types* of which only the *individual* is essential. The easiest method of defining an individual type is, again, by abstraction. The sub-class of a class is *individual* if its inductive implications are not shared with another sub-class. That is to say, if its properties are unique to itself. This notion of *individual* type, allows then the summation of classes of the same class (namely, number) by allowing them to be separated into individual sub-classes within a greater class. In this way, we can sum the class of 1, with another class of 1, and the result will be a class of two.

With this, it is now possible to assert logically that 1 + 1 = 2. Given that number has now been defined using only 4 simple axiomatic propositions and many logical assertions from those propositions. This undertaking of proving, as best as possible, the validity of axioms is very much in the spirit of Leibniz and is essential in any axiomatic system of which truth is the first and main proposition. From these simple assertions, however, it is possible to do more than merely prove that 1 summed with 1 is equal to 2; in fact, is possible to logically validate the existence of the entire set of rational numbers, the existence of infinity—indeed, we may (and have) established the existence of *degrees* of infinity from this!—and all of the rest of the arithmetic operations from which the whole of mathematics is derived.

Bibliography

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